

# Quantum Phenomenology with the Path Integral Approach

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## Abstract

A quantum measurement model based upon restricted path-integrals allows us to study measurements of generalized position in various one-dimensional systems of phenomenological interest. After a general overview of the method we discuss the cases of a harmonic oscillator, a bistable potential and two coupled systems, briefly illustrating their applications.

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Recent efforts in high precision experiments have shown that the fundamental limitations to a measurement due to quantum mechanics play a crucial role to develop more and more sensitive instruments [1]. As a by-product quantum measurement theory, a topic quite isolated from the frontier of physics some decades ago, has been revived and new experiments have been proposed for its better understanding. Models of quantum measurement theory comparable to the outcomes of the experiments are therefore welcome. Here we describe some of the results obtained using one of these models, originally developed in [2] (see also [3] for a complete account of the approach). The goal of this paper is twofold: to discuss specific examples of systems in which quantum measurement theory plays a crucial role and to show that they are all described within the *unified* context of the model we use. For reasons of space we do not deal here with a comparison of our approach to the other ones described elsewhere in these Proceedings [4]. Moreover, we remand the interested reader to the mentioned references of our papers for more details.

# A model for impulsive measurements

In classical mechanics the effect of a sequence of measurements on the subsequent dynamics is negligible. In quantum mechanics instead the execution of a measurement, either continuous or stroboscopic, influences the system. In this last case it is therefore important to include the effect of an actual measurement on the dynamics of the observed system. This has been obtained by Mensky [2, 3] through the path-integral approach (other similar approaches have been proposed in [4]). In this framework the propagator of a system described by a Lagrangian  $\mathcal{L}(x(t), \dot{x}(t), t)$ , and undergoing a continuous measurement of its position between the times 0 and  $\tau$ , with result  $a(t)$  and instrumental uncertainty  $\Delta a$ , is written as a weighted path-integral:

$$K_{[a]}(x'', \tau; x', 0) = \int_{x(0) \equiv x'}^{x(\tau) \equiv x''} \mathcal{D}[x(t)] \exp \left\{ \frac{i}{\hbar} \int_0^\tau \mathcal{L}(x(t), \dot{x}(t), t) dt \right\} w_{[a]}[x], \quad (1)$$

where

$$w_{[a]}[x] = \exp \left\{ -\frac{1}{2\Delta a^2 \tau} \int_0^\tau [x(t) - a(t)]^2 dt \right\}. \quad (2)$$

The most natural way to represent an impulsive measurement of position at time 0 is as limit of a continuous one for infinitesimal time intervals. In this approximation,

$$\lim_{\tau \rightarrow 0} K_{[a(t)]}(x'', \tau; x', 0) = K_a(x'', x') = e^{-\frac{(x' - a)^2}{2\Delta a^2}} K(x'', 0; x', 0) \quad (3)$$

where  $a = a(0)$ . Let us suppose that  $\psi(x, t)$  be the wavefunction of a system subjected to an impulsive measurement of position at time  $t$ , with result  $a$ . Since  $K(x'', 0; x', 0) = \delta(x'' - x')$ , it follows

$$\psi(x, t^+) = R_{[\psi]}(a) \int_{-\infty}^{+\infty} dy K_a(x, y) \psi(y, t^-) = R_{[\psi]}(a) \exp \left\{ -\frac{(\hat{x} - a)^2}{2\Delta a^2} \right\} \psi(x, t^-). \quad (4)$$

where  $R_{[\psi]}(a)$  is a renormalization constant. The effect of the measurement is therefore a Gaussian filtering around the measurement result. This is similar to the usual measurement theory of von Neumann. The latter is simply recovered by choosing for the measurement operator the discontinuous and therefore less realistic form

$$\hat{w}_a^{v.N.} \propto \theta(\hat{x} - [a - \Delta a]) \theta([a + \Delta a] - \hat{x}). \quad (5)$$

For a sequence of impulsive measurements the Gaussian reduction (4) is alternatively followed by free evolution periods  $\Delta T$  (quiescent time). The probability that the  $N^{th}$  measurement gives a result  $a_N$ , when the results of the previous  $N - 1$  ones are known, is expressed through

$$P_{a_1, a_2, \dots, a_{N-1}}(a_N) \equiv \frac{\left| \langle \psi_{a_1, a_2, \dots, a_N}(t_N^+) | \psi_{a_1, a_2, \dots, a_N}(t_N^+) \rangle \right|^2}{\int_{-\infty}^{+\infty} \left| \langle \psi_{a_1, a_2, \dots, a_N}(t_N^+) | \psi_{a_1, a_2, \dots, a_N}(t_N^+) \rangle \right|^2 da_N}. \quad (6)$$

which allows one to evaluate the effective uncertainty defined as

$$\Delta a_{eff}^2 = \frac{2 \int (a_N - \tilde{a})^2 P_{a_1, a_2, \dots, a_{N-1}}(a_N) da_N}{\int P_{a_1, a_2, \dots, a_{N-1}}(a_N) da_N}. \quad (7)$$

where  $\tilde{a}$  is the most probable measurement result. The effective uncertainty  $\Delta a_{eff}$  expresses the spreading of the possible measurement results [5] and is equal to  $\Delta a$ , the instrumental error, in the classical limit. In a quantum regime  $\Delta a_{eff}$  is always larger or equal to  $\Delta a$ . A convenient situation for evaluating the  $\Delta a_{eff}$  is available in the case of impulsive measurements when energy eigenstates and eigenvalues of the system are known. In such a case the evolution of the state during the quiescent time is obtained once an expansion in terms of energy eigenstates of the output of a measurement is made and the numerator of (6) can be rewritten as

$$\left| \langle \psi_{a_1, a_2, \dots, a_N}(t_N^+) | \psi_{a_1, a_2, \dots, a_N}(t_N^+) \rangle \right|^2 = \left( \sum_{m=1}^{\infty} |c_m^{(N)}|^2 \right)^2 \quad (8)$$

where  $c_m^{(N)}$  is the projection coefficient on the  $m^{th}$  eigenstate  $|m\rangle$  at the  $N^{th}$  measurement. Thus from the energy eigenstates expansion of the initial wavefunction it is possible to derive the effective uncertainty

$$\Delta a_{eff}^2 = \frac{2 \int_{-\infty}^{+\infty} (a_N - \tilde{a})^2 \left( \sum_{m=1}^{\infty} \left| \sum_{l=1}^{\infty} B_{ml}^N(a_0, \dots, a_{N-1}, a_N) c_l^{(0)} \right|^2 \right) da_N}{\int_{-\infty}^{+\infty} \left( \sum_{m=1}^{\infty} \left| \sum_{l=1}^{\infty} B_{ml}^N(a_0, \dots, a_{N-1}, a_N) c_l^{(0)} \right|^2 \right)^2 da_N}, \quad (9)$$

where

$$c_m^{(N)} \stackrel{\text{def}}{=} \sum_{l=1}^{\infty} B_{ml}^N c_l^{(0)}, \quad (10)$$

$$B_{ml}^N(\Delta T, \Delta a, a_1, a_2, \dots, a_N) \stackrel{\text{def}}{=} \langle m | \hat{w}_{a_N} \left( \prod_{j=1}^N e^{-\frac{i}{\hbar} \hat{H} \Delta T} \hat{w}_{a_{N-j}} \right) | l \rangle. \quad (11)$$

For quantum measurements on harmonic oscillators, having energy levels equally spaced, the optimal quiescent time is half the oscillation period and all can be evaluated without approximations. This is already known for the limit case of instantaneous and infinite accuracy measurements as seen using the canonical approach: the commutator for the position at different times gives [6]

$$[\hat{x}(t + \Delta T), \hat{x}(t)] = \frac{i\hbar}{m\omega} \sin \omega \Delta T \quad (12)$$

which implies that for two instantaneous measurements of position spaced by multiples of half period of oscillation the observable is quantum nondemolition. The path integral formalism allows us to extend such a result to finite accuracy and finite duration measurements, as shown in [5]. It turns out that the effective uncertainty

holds values very close to the instrumental uncertainty only for quiescent times which are multiples of half period of the harmonic oscillator. The optimal measurements of position for a harmonic oscillator are also known in literature as quantum nondemolition measurements [1, 6], and are quite important for the detection of small displacements like pulses of gravitational waves of astrophysical origin.

## Quantum measurements in SQuIDs

Superconducting Quantum Interferometer Devices (SQuIDs) have been proposed to test macrorealism versus quantum mechanics. A global understanding of quantum mechanics and its relationship to the classical limit requires to extend its validity to the macroscopic world. In this domain the conflict between its structure and the classical sense of physical reality cannot be overcome [7] and experiments aimed to compare the predictions of the two worldviews are crucial. By realism here we mean that “a macroscopic system with two or more macroscopically distinct states available to it will at all times *be* in one or the other of these states” (*macroscopic realism*) as well as that “it is possible, in principle, to determine the state of the system with arbitrarily small perturbation on its subsequent dynamics” (*non-invasive measurability at the macroscopic level*) [8]. This leads to a quantitative test by introducing inequalities between correlation functions of observables of a macroscopic system, a superconducting quantum interferometer device subjected to a sequence of repeated measurements of magnetic flux. The proposal of Leggett and Garg [8] has been criticized due to the limitations given by quantum mechanics to the accuracy obtainable in a set of repeated measurement of the same observable [9]. However, measurement schemes following that path have been proposed both using a set of SQuIDs and two-level mesoscopic systems [10]. We have studied quantitatively optimal strategies for repeated measurements of magnetic flux in bistable potentials which can schematize the SQuID behaviour [11]. It turns out that measurement strategies must be chosen in a particular way to minimize the influence of the previous measurements on the state of the observed system: quantum nondemolition strategies for the measurement of the magnetic flux in a SQuID have to be implemented. The magnetic flux in a SQuID is schematized, if the coupling to the external environment can be neglected, through the effective potential

$$V(\varphi) = -\frac{\mu}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 \quad (13)$$

where  $\varphi$  is the trapped magnetic flux, a generalized coordinate describing the system,  $\mu$  and  $\lambda$  are parameters associated to the superconducting circuit. This allows us to describe the system in terms of pure states  $\psi(\varphi)$ . Since general arguments exist on the fundamental noise introduced by any linear amplifier [12], the problem of the measurement of flux in a superconducting circuit is independent upon the detailed scheme used to detect the quantum state of the SQuID. As already discussed in [5] the optimality of the measurement is dictated by the spectral properties of the system. The asymptotic collapsed wavefunction can be expanded in terms of the eigenstates of the system and an optimal measurement is obtained provided that the

quiescent time is commensurable to the characteristic times for the wavefunction reformation

$$T_{ij} \equiv \frac{h}{|E_i - E_j|}, \quad (14)$$

where  $E_i$ ,  $E_j$  are the energy eigenvalues which have maximal projections on the asymptotic state. The two eigenstates which maximally contribute to the wavefunction reformation after tunneling are those corresponding to the first two eigenvalues, whose splitting is related to the tunneling period. The effective error  $\Delta\Phi_{eff}$  reached after a series of measurement pulses has minima close to  $\Delta\Phi$  when the quiescent time  $\Delta T$  is a multiple of the tunnelling period  $T_{12}$ . Unless the measurements are repeated with this periodicity noise due to the measurement process is fed into the system affecting the following measurements. In the case of the potential of Eq. (13) we have numerically evaluated the eigenstates using a selective relaxation algorithm [13]. The predictability of the measurement outcome is affected by the uncertainty due to the previous measurements. One can minimize this by simply using optimal quiescent times which are analytically determined through (14) once the eigenvalues and the eigenstates are given.

The general hypotheses under which we have obtained this result allows us to conclude that a direct experimental test of temporal Bell-type inequalities in principle should remain possible even in a pure quantum mechanical framework, without any classical assumption on the measurement process such as noninvasivity [8]. However, this happens only if the repetition times between consecutive measurements which violates Bell inequalities are compatible with the optimal quiescent times. Otherwise either the inequalities are not violated or quantum noise is introduced in the successive measurement, making it invasive [11]. A preliminary analysis on which we will refer in the future shows that this is the case. The region of violation occurs when the two quiescent times required to have a set of three possible measurements are both of the order of half of the tunnelling period. Unfortunately for such periodicities the effect of the measurement is such that the spreading on the possible results induced by the measurement does not allow one to distinguish between the two wells of the bistable potential. As a consequence violation of temporal Bell inequalities cannot be observed.

## Quantum measurements in Penning traps

Penning traps consist of a combination of static magnetic and electric fields where a single charged particle, for instance an electron, can be stored for long times [14]. A magnetic field confines the motion in a cylinder whose size is determined by the strenght of the magnetic field itself. A further confinement along the axis of the cylinder is obtained through a quadrupole electrostatic field creating a harmonic force parallel to the magnetic field. Thus the motion of the electron is a combination of a harmonic motion along the axis of the cylinder and a circular motion, called cyclotron motion, completely decoupled in the non-relativistic limit. They become coupled, with the spin too, by including relativistic corrections. In this case the

Hamiltonian of an electron in a Penning trap is

$$\begin{aligned}
H_{sys} = & \hbar\omega_c a_c^\dagger a_c \left\{ 1 - 1/2 \left( \frac{\omega_z}{\omega_c} \right)^2 - \frac{\hbar\omega_c}{2m_0 c^2} \right\} - \frac{(\hbar\omega_c)^2}{2m_0 c^2} (a_c^\dagger a_c)^2 + \\
& + \frac{p_z^2}{2m_0} + \frac{m_0 \omega_z^2 z^2}{2} - \frac{1}{2m_0 c^2} \left( \frac{p_z^2}{2m_0} \right)^2 + \frac{\hbar\omega_z \sigma_z}{4} \left[ g - \frac{\hbar\omega_c}{m_0 c^2} \left( 1 + \frac{g-1}{2} \left( \frac{\omega_z}{\omega_c} \right)^2 \right) \right] - \\
& - \frac{(\hbar\omega_c)^2}{2m_0 c^2} \left[ 1 + \frac{g-1}{2} \left( \frac{\omega_z}{\omega_c} \right)^2 \right] \sigma_z a_c^\dagger a_c - \frac{p_z^2}{2m_0^2 c^2} \hbar\omega_c a_c^\dagger a_c - g \frac{p_z^2}{8m_0^2 c^2} \hbar\omega_c \sigma_z \quad (15)
\end{aligned}$$

where  $a_c$  and  $a_c^\dagger$  are the annihilation and creation operators associated to the cyclotron motion,  $z$ ,  $p_z$  are position and momentum for the axial motion and  $\sigma_z$  is the projection of the spin on the  $z$ -axis. The interactions among the three degrees of freedom have small coupling constants proportional to  $c^{-2}$ .

Recently it has been proposed to measure the energy of the cyclotron motion of an electron in a Penning trap in a quantum nondemolition way [15]. A measurement of the energy associated to the cyclotron motion is obtained through the measurement of the axial motion of the trapped particle provided that the spin is preassigned. A detailed analysis of the dynamics of such a measurement is still missing, and the model discussed here can be applied to this situation. We note two peculiarities of such an application. Firstly, one is dealing with the case of non-quadratic terms in the Hamiltonian and numerical analysis is mandatory. Secondly, this is an example of an *indirect measurement*, in which the informations obtained on one degree of freedom allow one to measure some other quantities related to another degree of freedom coupled to the first (in this particular case the cyclotron energy through the direct measurement of spin and axial motion of the electron). The simplest prototype of systems in which indirect measurements are defined could consist of two coupled harmonic oscillators with only one subjected to measurement. In the particular case of position measurements the corresponding Hamiltonian could be written as

$$H_{eff} = \frac{p_1^2}{2m_1} + \frac{m_1 \omega_1^2 x_1^2}{2} + \gamma x_1 x_2 + \frac{p_2^2}{2m_2} + \frac{m_2 \omega_2^2 x_2^2}{2} + \frac{i\hbar}{\Delta a_1^2 \tau} [x_1(t) - a_1(t)]^2 \quad (16)$$

where the two oscillators are linearly coupled with strenght  $\gamma$ , and the measurement term directly affects only the first oscillator. It is easy to realize that, due to the coupling, a restriction of the paths in the coordinate space of the first oscillator will induce a restriction of the paths also in the second oscillator, therefore defining an indirect effective uncertainty  $\Delta a_{2eff}$ . A perturbative evaluation of such a quantity in the case of Eq. (16) is ongoing and we will refer on it in the future.

The interest of the above problem goes beyond the Penning traps application, and involves topics such as gravitational wave antennae coupled to a transducer [16] and quantum measurements of energy in microwave cavities [17].

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